The second major point we should highlight is that, according to eq. (80), when total utility is a sum of CRRA functions, the utility is additive, and each good/asset in the utility function behaves according to the CRRA type. The ratios of consumption across different goods (or different periods) depend only on their relative prices, not on income levels. This means that if there is an increase in real income, the consumption of all goods will go up exactly in the same proportion as income. In other words, all income elasticities are equal to one.

11 Exercises

In this section, we present a set of exercises that cover the main aspects of the RBC model, ranging from the crucial importance of intertemporal discounting to the accuracy of linearization, and we finish with changes in the model's main structure.

11.1 Exercise 1

We considered that the RBC model incorporates a subjective discount rate for future utility, denoted by d. The discount factor was defined as: $\beta = 1/(1+d)$. If we discount future utility at a rate of $d = 1\%$ per year, then $\beta \approx 0.99$. If $d = 3\%$, $\beta \approx 0.99$, and for $d = 5\%$ we will get $\beta \approx 0.952$.

Suppose we have two friends who visit Lisbon from time to time (Helen and Judy). Helen values the current life and consumption exactly as she values them in the future, even in the far future. This means that Helen's case will be $d = 0$ and $\beta = 1$. Judy, on the contrary, puts a lot of emphasis on her current life conditions and consumption, with a high discount rate of $d = 5\%$ per year on her future utility.

- **(a)** They go to a famous café in Lisbon and ask for an expresso coffee and a "Pastel de Belém". It is a nice combination that makes them feel good about life, and economists, giving an abstract representation of their joy of life, say that each one increased it by 1 *util*. Suppose they are considering the possibility of revisiting Lisbon in fifty years' time, and they want to come back to the same place and consume exactly the same goods. What is the today's utility value of that consumption bundle in fifty years' time, for both friends?
- **(b)** Abstracting from any other contingency factor that may affect their decisions, which friend will make a stronger case to consume those two goods in fifty years time? Explain.
- **(c)** In the case of the RBC model, what is the impact of an increase in the subjective discount rate d?

11.2 Exercise 2

However strange it may seem, discounting the future utility may be a perfectly rational and logical attitude. The RBC model does not incorporate growth factors and, for this reason, it is not the best tool for explaining the full rationale of this process. However, it can still shed some light on this issue. Consider the Euler equation that we have been discussing in this document, given by:

$$c_t^{-\sigma} = \beta \cdot c_{t+1}^{-\sigma} \cdot r_{t+1}$$

where r_{t+1} is defined as the net value of one unit of capital next period:

$$r_{t+1} \equiv \alpha \cdot a_{t+1} k_{t+1}^{\alpha-1} \ell_{t+1}^{1-\alpha} + 1 - \delta = MP_k + 1 - \delta \tag{E1}$$

that is: the sum of the marginal productivity of capital (MP_k) with 1, and subtracting the part of capital that is depreciated (δ). As we saw in Section 4, a steady-state to exist in this model requires that:

$$\overline{r} = \frac{1}{\beta}. \tag{E2}$$

As, by definition, we have that $\beta = 1/(1+d)$, we can easily obtain the following result in the steady state by combining (E1) and (E2):

$$\overline{MP}_k - \delta = \overline{d} \tag{E3}$$

Let us apply the result in (E3) to the case of Judy above. Suppose that today Judy has € 87.21 in her pockets. She decides not to spend them and instead invests this amount by buying Nvidia stock, for example. This company will increase its capital stock by using Judy's funds, and the value of that new capital will increase on average by 10% per year, with capital depreciating at 5% per year, over the next fifty years. Therefore, in fifty years, the value of Judy's investment will be equal to $r_{t+50} = 87.21(1 + 0.1 - 0.5)^{50} = 1000$.

(a) Considering this case, in your judgment, is the 5% subjective discount rate of future utility by Judy a rational economic decision? Explain.

(b) NVidia is currently a very special company with extremely above average profits due to the dramatic importance of its graphics cards. The average return on capital in an entire economy is certainly much lower than 10%. If so, is Judy's choice of a discount rate a good picture of the representative household we are considering in this document? Explain.

11.3 Exercise 3

To test the accuracy of the linearization processes discussed here, try a simple exercise. Assume the rate of change of a variable v is $g_v = 0.02$, or 2%, and another variable w grows at a rate $g_w = 0.02$. These numbers are typical for real-world macroeconomic variables. Also, suppose another variable μ behaves according to this function:

$$\mu_{t+1} = v_{t+1} \times w_{t+1} \tag{E4}$$

Question. What error do we make by using a linearized version of the function in (E1), compared to the actual function? (The solution is at the end of this section)

11.4 Exercise 4

Lets us test the accuracy of the linearization applied to a power function. Assume that the rate of change of a variable v is $g_v = 0.03$, that is 3%, which is very well in accordance with the numbers seen in the real world of macroeconomics. Consider also that another variable x behaves according to this function:

$$x_t = 4v_t^2 \tag{E5}$$

What is the error we make by using a linearized version of the function in (E2), when compared to the actual function? (There is a solution at the end of this section)

11.5 Exercise 5

In the standard version of the RBC model discussed in this document, we assumed that the coefficient of relative risk aversion with respect to labor supply, γ, is zero. This is a simplification used to render the derivation of the results easier. However, from a purely economic point of view, such an assumption does not make much sense, because the amount of hours we decide to offer during a period of time is not linear: we can not have the same marginal (dis)utility if we work one more hour per day, when we work 4 hours, or when we work 20 hours!

Therefore, in this exercise, we consider the case where $\gamma > 0$. The utility function that satisfies this condition is given by:

$$u(c_t, \ell_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \theta \frac{\ell_t^{1+\gamma}}{1+\gamma} \quad , \quad \{\sigma, \gamma, \theta\} \geq 0$$

This specification of the utility function produces only one change in the entire model we simulated in this document so far, which is the optimal labor supply, given by (31)

$$u'(\ell_t) = \frac{\partial u(c_t, \ell_t)}{\partial \ell_t} = -\theta \tag{30'}$$

The same optimal condition in this new version of the model will become:

$$u'(\ell_t) = \frac{\partial u(c_t, \ell_t)}{\partial \ell_t} = -\theta \cdot \ell_t^{\gamma}$$

which implies that the new eq. (31) will come out as:

$$\frac{y_t}{\ell_t^{1+\gamma}} = \left(\frac{\theta}{1-\alpha}\right) c_t^{\sigma}$$

The corresponding new linearized version of the previous equation will be as follows:

$$(1+\gamma)\hat{\ell}_t \approx \hat{y}_t - \sigma \hat{c}_t$$

The only change in this version of the model is that the labor supply will react much less to changes in output and in consumption, because the γ is greater than zero.

- **(a)** This is the only change to the model we discussed in this document. Create a new notebook to simulate this new version of the model and see what happens if, e.g., $\gamma = 1.5$.
- **(b)** Compare the main results of the model with the situation where $\gamma = 0$.
- **(c)** Does this new specification make the labor supply more (our less) reactive to the business cycles? Make a cross-plot of the changes in the labor supply $\hat{\ell}_t$ against changes in technology $\hat{a}_t$ and against changes in output $(\hat{yc}_t)$ and check with current data that can be seen in **?@fig-tpf_2**.

11.6 Exercise 6

The final exercise concerns another important point in the RBC model. Recall that in this model, there are no growth elements, which implies that there will be no growth in technology, in the total number of households in the economy, in the capital stock, or in real output. So, in such an environment, what will happen if, suddenly, there is a shock to the subjective discount rate, such that it becomes $d = 0.1$, leading to $\beta = 0.9$.

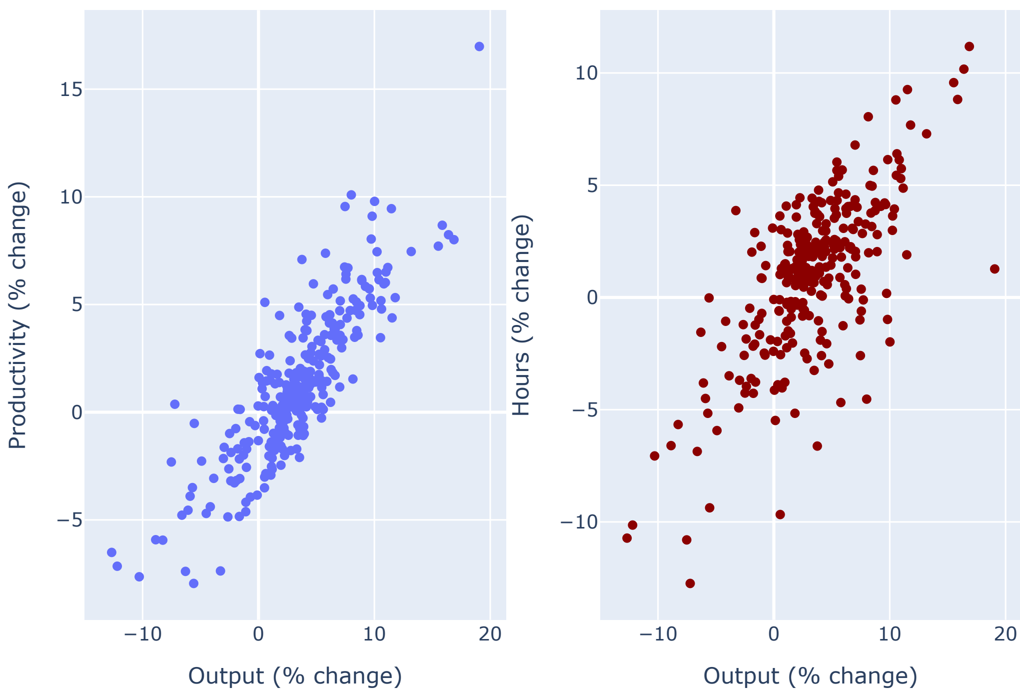

Figure 8: Output, total factor productivity, and labor supply

- **(a)** What will be the expected impact in the labor supply of such an increase in the discount rate?
- **(b)** Use a Pluto notebook and simulate that impact.

11.7 Solutions to exercises

Exercise 3. By using the rule (R2), the linearization of $\mu_{t+1} = v_{t+1} \times w_{t+1}$ will come out as:

$$g_\mu = g_v + g_w = 0.02 + 0.04 = 0.06.$$

To calculate the true rate of change of μ, we can implement a simple exercise with growth rates, knowing that by definition $\mu_{t+1} = (1 + g_\mu)\mu_t$, which implies that:

$$\frac{\mu_{t+1}}{\mu_t} = 1 + g_\mu$$

Therefore, by knowing that:

$$\mu_{t+1} = v_{t+1} \times w_{t+1}$$
$$\mu_t = v_t \times w_t$$

dividing μ_{t+1} by μ_t, leads to:

$$\underbrace{\left(\frac{\mu_{t+1}}{\mu_t}\right)}_{=1+g_\mu} = \frac{v_{t+1} \times w_{t+1}}{v_t \times w_t} = \frac{v_{t+1}}{v_t} \times \frac{w_{t+1}}{w_t} = (1 + g_v)(1 + g_w)$$
$$1 + g_\mu = (1 + g_v)(1 + g_w) = 1 + g_v + g_w + g_v g_w$$
$$g_\mu = g_v + g_w + g_v g_w$$
$$g_\mu = 0.02 + 0.04 + 0.02 \times 0.04 = 0.06 + 0.0008$$

Therefore, in the case of the linearization, we get $g_\mu = 0.06$, while in the case of the true nonlinear equation, we have $g_x = 0.06 + 0.0008$. So, the linearization error is only 0.0008, a figure that is hardly visible in national accounts, and we will not make any relevant mistakes if we stick to the linearization result.

Exercise 4. By using the rule (R3), the linearization of $x_t = 4v_t^2$ will come out as:

$$g_x = 2g_v = 2 \times 0.03 = 0.06.$$

To calculate the true rate of change of x, we can implement a simple exercise with growth rates, knowing that by definition $x_{t+1} = (1 + g_x)x_t$, which implies that:

$$\frac{x_{t+1}}{x_t} = 1 + g_x$$

Dividing x_{t+1} by x_t, and using the equation at the heart of this exercise $x_t = 4v_t^2$, we can write:

$$\underbrace{\left(\frac{x_{t+1}}{x_t}\right)}_{=1+g_x} = \frac{4v_{t+1}^2}{4v_t^2} = \underbrace{\left(\frac{v_{t+1}}{v_t}\right)^2}_{=1+g_v}$$

$$\begin{aligned}
1 + g_x &= (1 + g_v)^2 = 1 + 2g_v + g_v^2 \\
g_x &= 2g_v + g_v^2 \\
g_x &= 2 \times 0.03 + 0.03^2 \\
g_x &= 0.06 + 0.0006
\end{aligned}$$

Therefore, in the case of the linearization, we get $g_x = 0.06$, while in the case of the true nonlinear equation, we have $g_x = 0.06 + 0.0006$. So, the linearization error is only 0.0006, a very small number in terms of macroeconomic reality, and we will not make any relevant mistakes if we stick to the linearization result.

References

Arrow, Kenneth, Partha Dasgupta, Lawrence Goulder et al. (2004) "Are we consuming too much?" *Journal of economic perspectives*, 18 (3), 147–172.

Blanchard, Olivier Jean and Charles M Kahn (1980) "The solution of linear difference models under rational expectations," *Econometrica: Journal of the Econometric Society*, 1305–1311.

Blinder, Alan S (1988) "The fall and rise of Keynesian economics," *Economic record*, 64 (4), 278–294.

Christiano, Lawrence J and Martin Eichenbaum (1992) "Current real-business-cycle theories and aggregate labor-market fluctuations," *The American Economic Review*, 430–450.

Fernald, John (2014) "A quarterly, utilization-adjusted series on total factor productivity," Federal Reserve Bank of San Francisco.

Friedman, Milton (1968) "The role of monetary policy," *The American economic review*, 58 (1), 1–17.